Boundary element methods in probabilistic structural analysis (PBEM)

M. Ettouney, H. Benaroya and J. Wright

Applied Science Division, Weidlinger Associates, New York, NY, USA

Boundary Element Methods (BEM) are naturally suited for a wide field of engineering problems, especially those of a semi-infinite nature, for example, soil–structure interactions in earthquake or machine foundation problems. The required input parameters, dynamic loads, and system properties for the BEM are not, in general, very well defined and can be considered as random variables. Analysis methods for Probabilistic Finite Elements have been studied by many workers.[1] To the best knowledge of the authors, no such method is available for analysis of (Probabilistic) BEM and their uncertain input and/or system properties. In this work, we illustrate the usefulness of the probabilistic approach for engineering applications by using perturbation expansions to solve problems in soil modelling and structural analysis. In addition, the advantages of the probabilistic viewpoint are discussed with regard to current engineering practices. The importance of confidence estimates for criteria of non-exceedance response is emphasized.

Keywords: perturbation expansion, confidence estimates

Deterministic

Boundary element methods

The basic idea of Boundary Element Methods is to reduce a problem in n dimensions to one in $n - 1$ dimensions by using integral equation techniques to express the solution over a domain in terms of its surface values. This approach has been applied to problems in elastodynamics,[2] magnetoelasticity[3] and many other areas of study. We briefly describe its application to elastostatics problems; additional details can be found in Banerjee and Butterfield,[4] and Brebbia, et al.[5] A brief summary of integral equation formulations for elasticity problems is also given in Hong and Chen.[6]

Consider a linearly elastic body where $u_i(\xi)$ and $t_i(\xi)$ denote the components of displacement and traction, respectively, in the i-th direction at a point ξ in the body (*Figure 1*). For simplicity we assume that body forces are zero. Then the following integral equation, relating displacements and tractions, can be written

$$u_j(\xi) = \int_S [G_{ij}(x,\xi)t_i(x) - F_{ij}(x,\xi)u_i(x)]dS(x) \tag{1}$$

Dr. Benaroya is presently at the Department of Mechanical and Aerospace Engineering, Rutgers University, P.O. Box 909, Piscataway, NJ 08855–0909, USA.

Address reprint requests to Dr. Ettouney at the Applied Science Division of Weidlinger Associates, 333 Seventh Avenue, New York, NY 10001, USA.

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Integration in equation (1) is over all points on the surface S of the body, and the summation convention is used for repeated Latin subscripts. The kernel functions are known free-space Green's functions (fundamental solutions). For example, $G_{ij}(x,\xi)$ is the displacement in the j-th direction at point ξ in the body due to a unit traction in the i-th direction at surface point x. $F_{ij}(x,\xi)$ is the respective fundamental deterministic singular solution of surface tractions. In the case of two-dimensional, plane strain elasticity, we have the displacement Green's function

$$G_{ij}(x,\xi) = C_1\left(C_2\delta_{ij}\ln r - \frac{y_iy_j}{r^2}\right) + A_{ij} \tag{2}$$

where:
$$\begin{aligned}
C_1 &= -\frac{1}{8\pi\mu(1-\nu)}\\
C_2 &= 3 - 4\nu\\
\mu &= \text{shear modulus}\\
\nu &= \text{Poisson's ratio}\\
A_{ij} &= \text{arbitrary constant tensor}\\
y_i &= x_i - \xi_i\\
y_j &= x_j - \xi_j\\
r^2 &= y_iy_i\\
\delta_{ij} &= \text{Dirac delta function}\\
i,j &= 1,2 \text{ for plane strain case}
\end{aligned}$$

and the traction Green's function

$$F_{ik}(x,\xi) = \frac{C_3}{r^2}\left[C_4(n_ky_i - n_iy_k) + \left(C_4\delta_{ik} + \frac{2y_iy_k}{r^2}\right)y_jn_j\right] \tag{3}$$

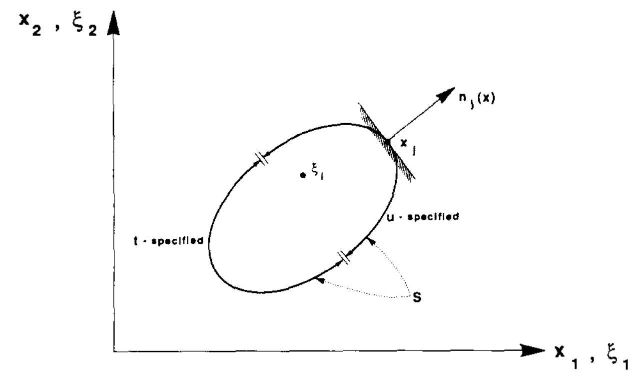

Figure 1. Linear elastic body

where: $C_3 = -\dfrac{1}{4\pi(1 - \nu)}$
$C_4 = 1 - 2\nu$
$n_j(x)$ = outward normal at x

Once surface values are properly defined, the problem specification is complete. For simplicity we assume that displacements are specified over certain portions of the surface and that tractions are specified over the remaining portions.

In order to simplify some of the subsequent computations, equation (1) is rewritten as

$$\int_S \mathbf{G}t dS = \int_S \mathbf{F}u dS + (-4\pi)Qu \tag{4}$$

where $Q = \mu(1 - \nu)$, and matrices **G** and **F** have been suitably redefined to account for the factor on μ. Equation (4) is used in the probabilistic analysis.

A numerical solution can be obtained by discretizing the surface and approximating the integrals in equation (4) by a weighted sum of functions evaluated at the surface points. If m is the total number of points on the surface, then equation (4) is approximated by a set of linear algebraic equations with order proportional to m (roughly $2m$ for two dimensions, $3m$ for three dimensions):

$$\mathrm{M}p = \mathrm{N}q \tag{5}$$

M and N are square matrices whose elements are computed by numerical integration of the kernel functions with due regard being given to integrable singularities. p and q are vectors whose entries are the surface displacements and tractions with p containing the unknown values and q the specified values. Solution of these equations yields approximate values for displacements over the portions of the surface where tractions are specified, and approximate values for tractions where displacements are specified. Once tractions and displacements are known on the entire surface, displacements anywhere in the body can be calculated from the discretized form of equation (4). Other quantities, such as strains, stresses and tractions in the body, can be computed easily from similar approximations once the surface values have been determined.

Probabilistic

Perturbation expansions and uncertainty modelling

In general, uncertainties can arise in material properties, specified surface values, the geometry of the body, or any combination of these. In extending the formulation to include uncertainties, we distinguish between various classes of problems. The first distinction is between deterministic and random systems. The concept of randomness or uncertainty may be distinguished as follows:

1. uncertainty regarding constant system parameters results in a "spread" or density of possible parameter values, i.e., random variable model,
2. uncertainty regarding the behavior of parameters as time-dependent processes, where the statistics of each process may be either time-dependent or -independent, i.e., random or stochastic function model, and
3. uncertainty with respect to the behavior of parameter properties as functions of spatial coordinates where, again, the statistics of the functional behavior may be either space-dependent or -independent, i.e., random or stochastic field model.

Except for the case where only surface values are random, these are, in general, all very difficult problems to solve, both mathematically and in application to existing problems. We are initially interested in the random variable model, since this will be of most practical value for the applications in mind. Extensions of the developed ideas will be possible subsequently.

A case of considerable interest in engineering applications is one where the statistics of the material parameters μ and ν are available so that their mean values are known and deviations relative to the mean are not too large. We consider this case and limit the problem to a body of one material with given surface values and geometry. Then the unknown displacements and tractions on the surface can be expanded about the mean values in a Taylor series.

Perturbation expansions

The dependence on μ and ν in equation (5) can be shown explicitly by writing

$$\mathbf{M}(\mu,\nu)p(\mu,\nu) = \mathbf{N}(\mu,\nu)q \tag{6}$$

where q is specified, and therefore has no μ, ν dependence. Expansions in Taylor series about the mean values $\bar{\mu}$ and $\bar{\nu}$, with $\epsilon_\mu = (\mu - \bar{\mu})$ and $\epsilon_\nu = (\nu - \bar{\nu})$, yields

$$p(\mu,\nu) = \bar{p} + \epsilon_\mu \bar{p}_\mu + \epsilon_\nu \bar{p}_\nu + \tfrac{1}{2}\epsilon_\mu^2 \bar{p}_{\mu\mu} + \tfrac{1}{2}\epsilon_\nu^2 \bar{p}_{\nu\nu} + \epsilon_\mu \epsilon_\nu \bar{p}_{\mu\nu} + \cdots \tag{7}$$

where the subscripts on $\bar{p}$ denote partial differentiation (no summation implied for repeated Greek subscripts) and the overline denotes evaluation of the functions at the respective mean values. Similar notation will be used for Taylor series expansions of $\mathbf{M}(\mu,\nu)$, $\mathbf{N}(\mu,\nu)$ and other quantities below. As discussed after equation

(4), some simplification is obtained by first clearing any μ and ν dependence from the denominators of both sides of equation (6). The result is that **M** and **N** have terms which are proportional only to $1,\mu,\nu$ or $\mu\nu$, thereby producing truncated Taylor series for these quantities. Substitution into equation (6), and equating terms of the same order in ϵ_μ and ϵ_ν, produces the usual set of recursion relations which result from perturbation expansion techniques:[7]

$$\overline{\mathbf{M}}\bar{p} = \overline{\mathbf{N}}q \tag{8a}$$

$$\overline{\mathbf{M}}\bar{p}_\nu = \overline{\mathbf{N}}_\nu q - \overline{\mathbf{M}}_\nu\bar{p} \tag{8b}$$

$$\overline{\mathbf{M}}\bar{p}_\mu = \overline{\mathbf{N}}_\mu q - \overline{\mathbf{M}}_\mu\bar{p} \tag{8c}$$

$$\overline{\mathbf{M}}\bar{p}_{\mu\mu} = -2\overline{\mathbf{M}}_\mu\bar{p}_\mu \tag{8d}$$

$$\cdots$$

The recursive system provides an efficient procedure for calculating the terms in equation (7) since the coefficient matrix $\overline{\mathbf{M}}$ is the same for all orders. Once the elements of this matrix have been computed and the matrix has been factorized into lower and upper triangular matrices, most of the computational work has been done (order m^3 operations), and the equations can be solved quickly (order m^2 operations). It is relatively easy to assess the rate of convergence of the series and, as expected, the expansion converges rapidly for physically reasonable uncertainties in the values of μ and ν, except for $\bar{\nu}$ close to 0.5, which is a point of singularity for the elasticity problem.

Specifically, with reference to equation (4)

$$u(\mu,\nu) = \bar{u} + \epsilon_\mu\bar{u}_\mu + \epsilon_\nu\bar{u}_\nu + \tfrac{1}{2}\epsilon_\mu^2\bar{u}_{\mu\mu} + \tfrac{1}{2}\epsilon_\nu^2\bar{u}_{\nu\nu} + \epsilon_\mu\epsilon_\nu\bar{u}_{\mu\nu} + \cdots \tag{9a}$$

$$t(\mu,\nu) = \bar{t} + \epsilon_\mu\bar{t}_\mu + \epsilon_\nu\bar{t}_\nu + \tfrac{1}{2}\epsilon_\mu^2\bar{t}_{\mu\mu} + \tfrac{1}{2}\epsilon_\nu^2\bar{t}_{\nu\nu} + \epsilon_\mu\epsilon_\nu\bar{t}_{\mu\nu} + \cdots \tag{9b}$$

$$\mathbf{G}(\mu,\nu) = \overline{\mathbf{G}} + \epsilon_\nu\overline{\mathbf{G}}_\nu \tag{9c}$$

$$\mathbf{F}(\mu,\nu) = \overline{\mathbf{F}} + \epsilon_\nu\overline{\mathbf{F}}_\nu + \epsilon_\mu\overline{\mathbf{F}}_\mu + \epsilon_\mu\epsilon_\nu\overline{\mathbf{F}}_{\mu\nu} \tag{9d}$$

$$-4\pi Q = -4\pi[\bar{Q} + \epsilon_\mu\bar{Q}_\mu + \epsilon_\nu\bar{Q}_\nu + \epsilon_\mu\epsilon_\nu\bar{Q}_{\mu\nu}] \tag{9e}$$

represent the Taylor series expansions of functions μ, t, **G**, **F**, and Q in terms of the respective μ and ν "sensitivity functions". Since six terms in the series are retained (i.e., expansions up to second order terms in ϵ), six equations corresponding to (4) must be solved

$$\int_S \overline{\mathbf{G}}\,\bar{t}\,dS = \int_S \overline{\mathbf{F}}\bar{u}\,dS + (-4\pi)\bar{Q}\bar{u} \tag{10a}$$

$$\int_S (\overline{\mathbf{G}}\,\bar{t}_\mu)dS = \int_S (\overline{\mathbf{F}}\bar{u}_\mu + \overline{\mathbf{F}}_\mu\bar{u})dS + (-4\pi)[\bar{Q}_\mu\bar{u} + \bar{Q}\bar{u}_\mu] \tag{10b}$$

$$\int_S (\overline{\mathbf{G}}\,\bar{t}_\nu + \overline{\mathbf{G}}_\nu\bar{t})dS = \int_S (\overline{\mathbf{F}}\bar{u}_\nu + \overline{\mathbf{F}}_\nu\bar{u})dS + (-4\pi)[\bar{Q}_\nu\bar{u} + \bar{Q}\bar{u}_\nu] \tag{10c}$$

$$\int_S (\overline{\mathbf{G}}\,\bar{t}_{\mu\mu})dS = \int_S (\overline{\mathbf{F}}\bar{u}_{\mu\mu} + 2\overline{\mathbf{F}}_\mu\bar{u}_\mu)dS + (-4\pi)[2\bar{Q}_\mu\bar{u}_\mu + \bar{Q}\bar{u}_{\mu\mu}] \tag{10d}$$

$$\int_S (\overline{\mathbf{G}}\,\bar{t}_{\nu\nu} + 2\overline{\mathbf{G}}_\nu\bar{t}_\nu)dS = \int_S (\overline{\mathbf{F}}\bar{u}_{\nu\nu} + 2\overline{\mathbf{F}}_\nu\bar{u}_\nu)dS + (-4\pi)[2\bar{Q}_\nu\bar{u}_\nu + \bar{Q}\bar{u}_{\nu\nu}] \tag{10e}$$

$$\int_S (\overline{\mathbf{G}}\,\bar{t}_{\mu\nu})dS = \int_S (\overline{\mathbf{F}}\bar{u}_{\mu\nu} + \overline{\mathbf{F}}_\nu\bar{u}_\mu + \overline{\mathbf{F}}_\mu\bar{u}_\nu + \overline{\mathbf{F}}_{\nu\mu}\bar{u})dS + (-4\pi)[\bar{Q}\bar{u}_{\mu\nu} + \bar{Q}_\nu\bar{u}_\mu + \bar{Q}_\mu\bar{u}_\nu + \bar{Q}_{\mu\nu}\bar{u}] \tag{10f}$$

Equations (10a–f) are solved recursively for the unknown components of $\bar{u}$ and $\bar{t}$, $\bar{u}_\mu$ and $\bar{t}_\mu$, $\bar{u}_\nu$ and $\bar{t}_\nu$, $\bar{u}_{\mu\mu}$ and $\bar{t}_{\mu\mu}$, $\bar{u}_{\nu\nu}$ and $\bar{t}_{\nu\nu}$, $\bar{u}_{\mu\nu}$ and $\bar{t}_{\mu\nu}$, and substituted appropriately into equations (9a,b), which are subsequently (below) used to evaluate the first and second order moments (mean and standard deviation) of u and t.

Two random variables

Equation (9a,b) provide the basis for the probabilistic analysis of the response of an elastic body. The mean values of the displacements and tractions, respectively, are approximately given by

$$E\{u(\mu,\nu)\} \cong \bar{u} + E\{\epsilon_\mu\}\bar{u}_\mu + E\{\epsilon_\nu\}\bar{u}_\nu + \tfrac{1}{2}E\{\epsilon_\mu^2\}\bar{u}_{\mu\mu} + \tfrac{1}{2}E\{\epsilon_\nu^2\}\bar{u}_{\nu\nu} + E\{\epsilon_\mu\epsilon_\nu\}\bar{u}_{\mu\nu} \tag{11a}$$

$$E\{t(\mu,\nu)\} \cong \bar{t} + E\{\epsilon_\mu\}\bar{t}_\mu + E\{\epsilon_\nu\}\bar{t}_\nu + \tfrac{1}{2}E\{\epsilon_\mu^2\}\bar{t}_{\mu\mu} + \tfrac{1}{2}E\{\epsilon_\nu^2\}\bar{t}_{\nu\nu} + E\{\epsilon_\mu\epsilon_\nu\}\bar{t}_{\mu\nu} \tag{11b}$$

and the mean-square values, correct to second order, by

$$E\{u(\mu,\nu)^2\} \cong \bar{u}^2 + E\{\epsilon_\mu^2\}(\bar{u}\,\bar{u}_{\mu\mu} + \bar{u}_\mu^2) + E\{\epsilon_\nu^2\}(\bar{u}\,\bar{u}_{\nu\nu} + \bar{u}_\nu^2) \tag{12a}$$

$$E\{t(\mu,\nu)^2\} \cong \bar{t}^2 + E\{\epsilon_\mu^2\}(\bar{t}\,\bar{t}_{\mu\mu} + \bar{t}_\mu^2) + E\{\epsilon_\nu^2\}(\bar{t}\,\bar{t}_{\nu\nu} + \bar{t}_\nu^2) \tag{12b}$$

where it is recalled that $\epsilon_\mu = (\mu - \bar{\mu})$ and $\epsilon_\nu = (\nu - \bar{\nu})$, and thus $E\{\epsilon_\mu\} = 0$ and $E\{\epsilon_\nu\} = 0$. The mathematical expectation of order n for a random variable x is defined as

$$E\{x^n\} = \int_{-\infty}^{\infty} x^n f_x(x)\, dx \tag{13}$$

For $n = 1$, equation (13) provides the expected or mean value of x, and for $n = 2$ the mean-square value. In order to evaluate these expressions, we need to make probabilistic statements regarding the random variables μ and ν, i.e., probability densities $f_\mu(\mu)$ and $f_\nu(\nu)$. Of course, the upper and lower limits in equation (13) will reflect the actual range on the respective parameters.

Four possible probability models are chosen to represent the random variables μ and ν:

1. the **uniform** density[8]

$$f_\mu(\mu) = \begin{cases} \dfrac{1}{\mu_2 - \mu_1}, & \mu_1 \le \mu \le \mu_2 \\ 0, & \text{otherwise} \end{cases} \tag{14a}$$

$$f_\nu(\nu) = \begin{cases} \dfrac{1}{\nu_2 - \nu_1}, & \nu_1 \le \nu \le \nu_2 \\ 0, & \text{otherwise} \end{cases} \tag{14b}$$

2. the **symmetric truncated normal** density[9]

$$f_\mu(\mu) = \begin{cases} \dfrac{A}{\sigma\sqrt{2\pi}} exp\left[-\dfrac{(\mu - \bar{\mu})^2}{2\sigma^2}\right], & \mu_1 \le \mu \le \mu_2 \\ 0, & \text{otherwise} \end{cases} \tag{15a}$$

$$f_\nu(\nu) = \begin{cases} \dfrac{A}{\sigma\sqrt{2\pi}} exp\left[-\dfrac{(\nu - \bar{\nu})^2}{2\sigma^2}\right], & \nu_1 \le \nu \le \nu_2 \\ 0, & \text{otherwise} \end{cases} \tag{15b}$$

$$A = \frac{1}{2\,\text{erf}(k)}$$

$$\text{erf}(k) = \frac{1}{\sqrt{2\pi}} \int_0^k exp\left[-\frac{y^2}{2}\right] dy$$

for the cases $k = 1, 2, 3$, where $k\sigma_N$ equals one-half the difference between the upper and lower bounds defined above, and σ_N is the standard deviation of the equivalent non-truncated normal density. Thus, $k = 1$ is the normal density truncated symmetrically about $\pm\sigma$; $k = 2$ is the normal density truncated symmetrically about $\pm 2\sigma$; $k = 3$ is the normal density truncated symmetrically about $\pm 3\sigma$.

For the uniform density, $E\{\epsilon_\mu \epsilon_\nu\} = E\{\epsilon_\mu\}E\{\epsilon_\nu\}$, i.e., the variables are uncorrelated, and thus the expression is identically zero. This property is not generally the case for the truncated normal (or any other) density, but we assume uncorrelatedness for all examples studied here for lack of any specific data. Given such information, it would be possible to estimate correlation.

Once equations (11) and (12), in conjunction with (13), are used to approximate $E\{u(\mu,\nu)\}$, $E\{t(\mu,\nu)\}$, $E\{u(\mu,\nu)^2\}$, and $E\{t(\mu,\nu)^2\}$, it is useful to evaluate the respective variances (σ^2) and standard deviations (σ) of displacements and tractions

$$\sigma_u = +\sqrt{\sigma_u^2} = \sqrt{E\{u^2\} - E^2\{u\}} \tag{16a}$$

$$\sigma_t = +\sqrt{\sigma_t^2} = \sqrt{E\{t^2\} - E^2\{t\}} \tag{16b}$$

Given the mean and variance, it is possible to establish confidence bounds on specific realizations of u and t. One possible approach, for a random variable x with mean value η, is to use the Chebyshev inequality

$$P\{\eta - a < x < \eta + a\} \ge 1 - \frac{\sigma^2}{a^2} \tag{17}$$

or the one-sided versions

$$P\{x < \eta + a\} \ge \frac{a^2}{\sigma^2 + a^2} \tag{18a}$$

$$P\{x > \eta - a\} \ge \frac{a^2}{\sigma^2 + a^2} \tag{18b}$$

Specific applications are introduced and discussed in the following section.

Applications

The theoretical PBEM equations were introduced in a previous section, where we also discussed the behavior of plane elastic systems with uncertainties in the material properties. The purpose of this section is to study some of the possible practical applications of PBEM, and to correlate between this method and present engineering practices.

Current practices

Different engineers, analysts and designers are well aware of the uncertainties of material properties which were discussed earlier. One way to account for the uncertainties is to perform an expensive and time consuming Monte Carlo simulation of the system at hand, which is not exactly a practical solution of the problem. Another approach is to assess the uncertainties at three levels using deterministic approaches. First the system response is evaluated using the "best guess" of the material properties. It is then evaluated using a combination of upper, and/or, lower limits of the material properties in order to produce upper and lower bound values of the response quantities of interest. The space between the upper and lower bounds (which will contain the "best guess" point) can be called the "deterministic zone". The deterministic zone can then be used for any design or analysis decision.

The deterministic zone approach, although practical, has several drawbacks. First, it does not provide any information about the behavior of the response parameters of interest outside the zone. It can be at times too conservative, at other times not, and the designer/analyst has no way of knowing which. Confidence estimates are out of the question, i.e., what are the probabilities that response-parameter values will fall outside the deterministic zone?

The PBEM, and similar methods, can solve this dilemma for the practicing engineer in an easy and accurate way. In what follows are presented some examples of the application of the PBEM to the fields of structural and geotechnical engineering. One basic and important problem in each field is studied using the PBEM; it is then compared with the deterministic zone results, with discussion.

Beam stiffness

Uncertainty in elastic beam properties

Linear elastic analysis of beams requires the definition of two independent material properties, usually μ and ν (the shear modulus and Poisson's ratio), respectively. For some engineering materials such as steel, these properties are well defined, with little or no uncertainty in the definition of magnitudes. However, there exist several situations where a precise definition of μ and ν is impossible. This leads to a great difficulty in the analysis process. Examples of such situations are:

1. Reinforced (or plain) concrete beams: Poisson's ratio of concrete beams is not well defined at all. Kong et al. recommend a range for Poisson's ratio of 0.0 to 0.20, depending on the level of cracking in the concrete.[10] Similar uncertainty exists for the definition of the shear modulus, where a range of values is usually recommended in order to account for different uncertainties in the concrete behavior.
2. Equivalent (built up) beams: Structural engineers usually use an equivalent beam model to simulate a system with several components (*Figure 2*). In these situations, only an estimate of the equivalent material properties of an equivalent beam is possible, even if the material properties of each of the components of the system is accurately defined.

It is of interest to investigate the sensitivity of beam behavior to the variability of its material properties. The methods outlined earlier in this study will be used to do this. Although the analytical solutions of beams with a long span to depth ratio is widely available,[11] the numerical solutions of the PBEM will be needed to obtain the solutions for shorter beams.

Stiffness in shear mode

Consider the beam of *Figure 3a*. It has a square cross section with unit dimension. The length of the beam is L. One of the best ways to study the sensitivity of the behavior of this beam is to study its stiffnesses in a shear deformation mode, *Figure 3b*, since these stiffnesses are used directly in the beam stiffness ma-

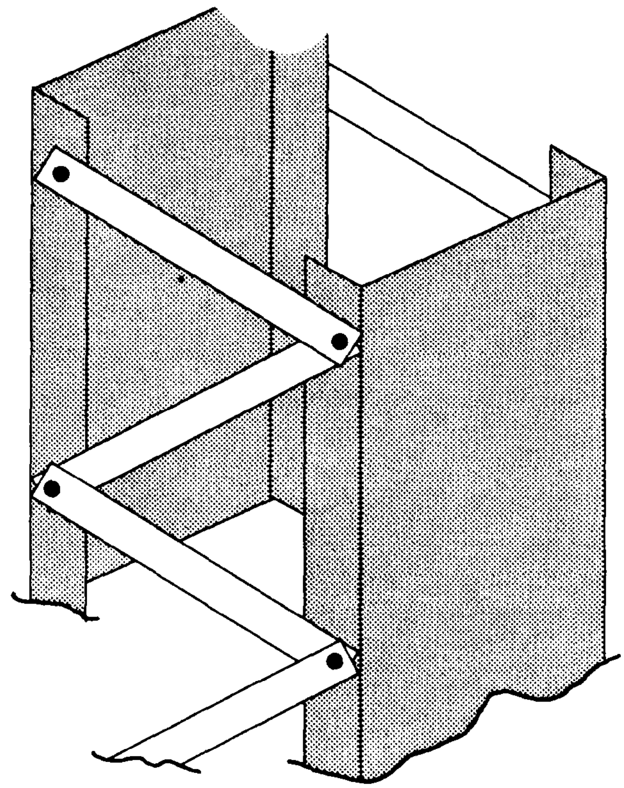

Figure 2. Built-up beams

trix in most practical engineering applications. (Bending and axial deformation modes, though as important as the shear deformation mode, are not included in this study due to time and space limitations.)

A boundary element model prepared for the beam under consideration is represented in *Figure 3c*. The accuracy of this discretization is judged to be adequate for the purposes of this study. The surface tractions are assumed to be zero at the top and bottom surfaces of the beam. The front and back ends of the beam are restrained in the axial direction, while the relative vertical displacements between the two ends of the beam are assumed to be unity. The shear stiffness, K_{ss}, is calculated by integrating the shear tractions along one of the ends of the beam. The coupled bending/shear stiffness, K_{sb}, is calculated by evaluating the resultant bending moment about the center of the beam at the beam support.*

The material properties of this beam are assumed to be random variables. *Table 1* shows the expected values, as well as the upper and lower limits of both Poisson's ratio and the shear modulus. These numbers are chosen only for study purposes, although it is believed that these ranges of variability do represent realistic situations.

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* Note that $K_{ss} = \frac{12EI}{L^3}$ and $K_{sb} = \frac{6EI}{L^2}$ for long beams, where E and I are the modulus of elasticity and cross sectional moment of inertia, respectively

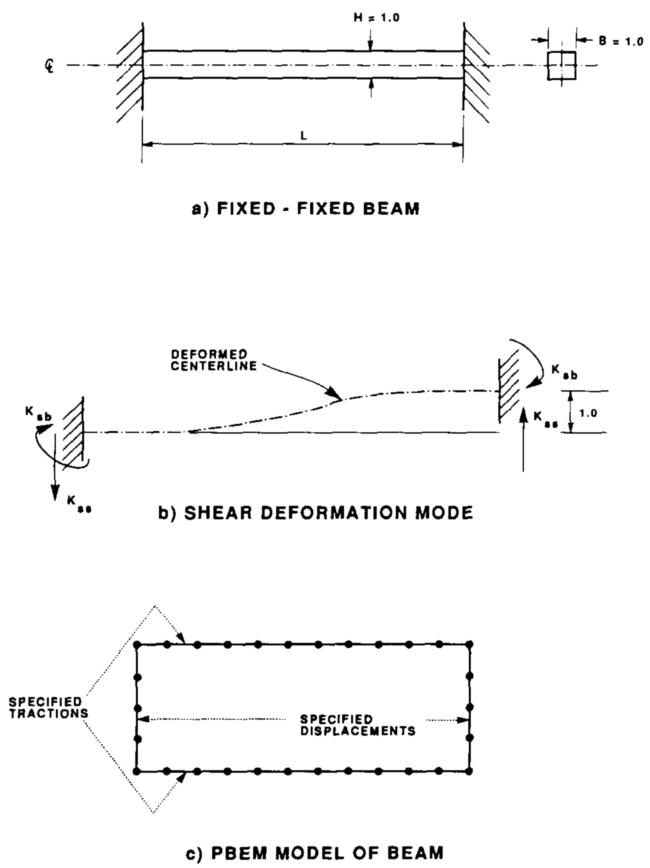

Figure 3. Structural beam: (a) fixed–fixed beam, (b) shear deformation mode, (c) PBEM model of beam

Table 1. Beam material properties

$\bar{\mu}$	μ_1	μ_2	$\bar{\nu}$	ν_1	ν_2
1.0	0.75	1.25	0.3	0.2	0.4

Several cases for the beam stiffnesses are studied. These cases are for $L = 5$, 3, 1 and 0.5. Four probability density functions of the material properties are used: the uniform density, and the truncated Gaussian density with $k = 1$, 2 and 3. In each case, the expected value and the standard deviation of the stiffnesses are evaluated. The maximum and minimum stiffnesses with a probability of non-exceedance of 84% and 97% are calculated using the Chebyshev inequalities which were described earlier. These non-exceedance levels are chosen because they correspond to a one and two standard deviation from the mean in a normal density function; hence, they are widely used in engineering applications.

In order to gain more insight to the "probabilistic" stiffnesses, the deterministic zones for the stiffnesses corresponding to *Table 1* are calculated.

Finally, all the probabilistic and deterministic stiffnesses have been normalized with respect to the stiffness calculated deterministically using the mean values, i.e., "best guess", of the material properties.

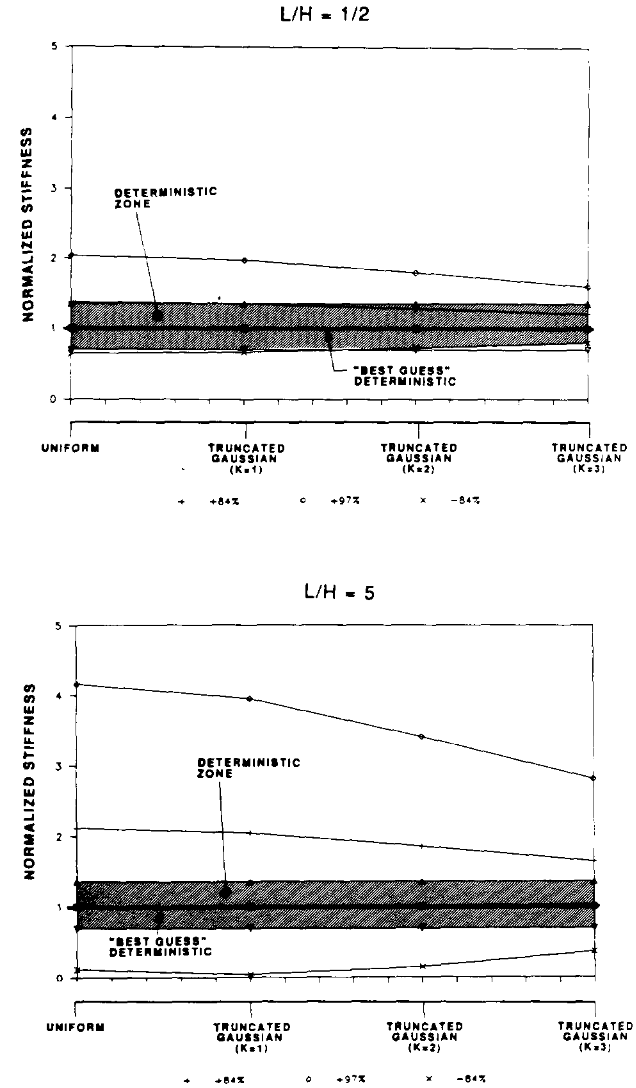

Figure 4. Shear stiffness, K_{ss}

Figure 4 shows the normalized shear stiffness, K^0_{ss}, for $L/H = \frac{1}{2}$,5. The stiffnesses with probabilities of non-exceedance of 84% and 97% are always out of the bounding deterministic zone stiffnesses for the uniformly distributed material properties. As expected, the truncated Gaussian densities with larger k result in probabilistic stiffnesses closer to the deterministically calculated mean and bounds.

It is of interest to note that the requirements for the stiffnesses for the two non-exceedance levels get to be more severe for longer beams. This is due to the fact that the stiffnesses for long beams are direct functions of the modulus of elasticity, E, which is a function of the two random variables μ and ν through the relation $E = 2\mu(1 + \nu)$. For shorter beams, on the other hand, the shear stiffness becomes more dependent on the shear modulus μ and less dependent on the modulus of elasticity E or Poisson's ratio ν. In other words, the shear stiffness for long beams is a function of two random variables (μ and ν) whereas the shear stiffness

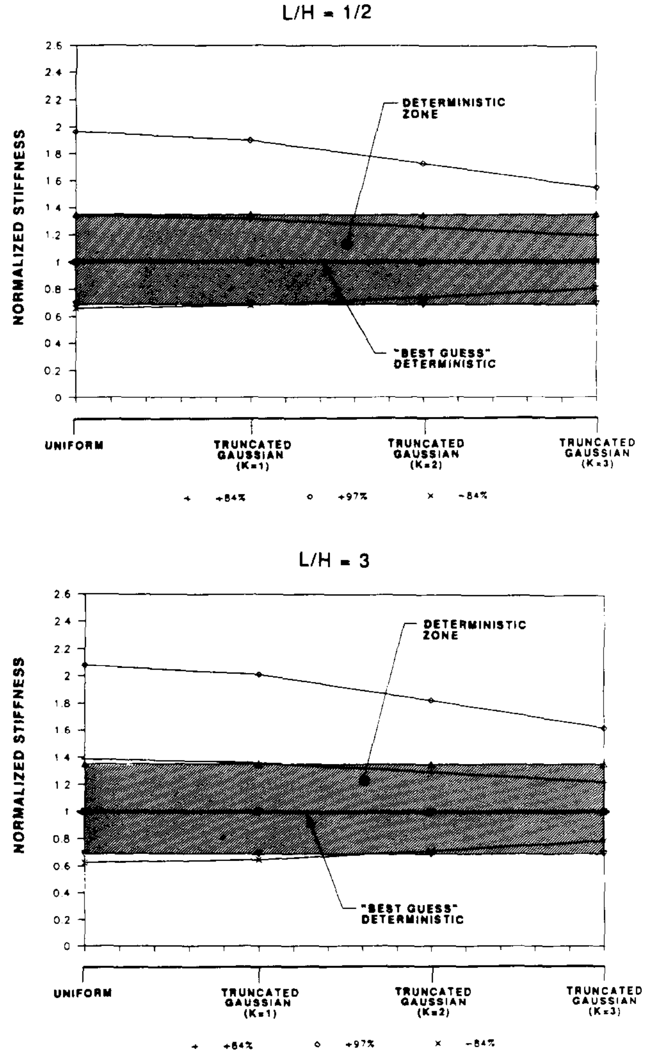

Figure 5. Shear/bending stiffness, K_{sb}

for short beams is a function of one random variable (μ).

Figure 5 shows similar results for the normalized shear/bending stiffness, again for $L/H = \frac{1}{2}, 5$. The behavior is different from that of the shear stiffness. First, the boundaries of the deterministic zones are of the same magnitude as that of the stiffness with 84% non-exceedance probability. Only higher probability levels get outside the bounding levels. Second, the behavior of the normalized stiffness seems to be insensitive to the length of the beam. This is due to the fact that the bending moment for shorter beams is dependent on both μ and ν.

Table 2. Soil properties

Soil case	$\bar{\mu}$	μ_1	μ_2	$\bar{\nu}$	ν_1	ν_2
I	1	0.5	1.5	0.3	0.26	0.34
II	1	0.5	1.5	0.35	0.31	0.39
III	1	0.5	1.5	0.4	0.36	0.44
IV	1	0.5	1.5	0.45	0.41	0.49

Static soil stiffness

Soil stiffnesses are important parameters in the fields of soil dynamics, foundations and earthquake engineering. They are also the cornerstone parameters in the field of machine foundations. Soil stiffnesses are, of course, frequency dependent;[12] however, Kausel[13] has shown that the static soil stiffness can play a major role in the definition of the frequency dependent soil stiffnesses. We will study the plane static soil stiffness of a half-space, and a soil layer with a finite depth in what follows.

Uncertainty in elastic soil properties

Linear elastic analysis of soils involves the use of two material properties, generally Poisson's ratio and the shear modulus. For the purpose of engineering projects, these properties are usually evaluated on a "best guess" and "range" basis. For the purposes of this paper, we study four material cases, as shown in *Table 2*. These cases are chosen to represent practical situations. The main difference between the four soils is that the "best guess", or the expected value, of Poisson's ratio are taken as 0.30, 0.35, 0.40 and 0.45. Note that the uncertainty range of Poisson's ratio is kept constant for all the cases under consideration for better comparison. The expected values, as well as the range of the shear modulus, are also kept constant for all cases.

The soil stiffnesses with probability of non-exceedance of 84% and 97% are calculated using the PBEM of this paper. The material properties are assumed to be random variables with the same probability density functions as that of the structural beam section. The soil stiffnesses are also calculated deterministically using the expected value of the material properties. The deterministic zones are then calculated using the approximate analytical expressions developed by Jacob.[14] All the results are then normalized with respect to the deterministic "best guess" value.

Half-space

The plane rigid massless footing on an elastic half-space (*Figure 6*) will be considered next. *Figure 7* shows, for cases I and IV, the different results for the rocking stiffness of the rigid footing on a half-space. The effect

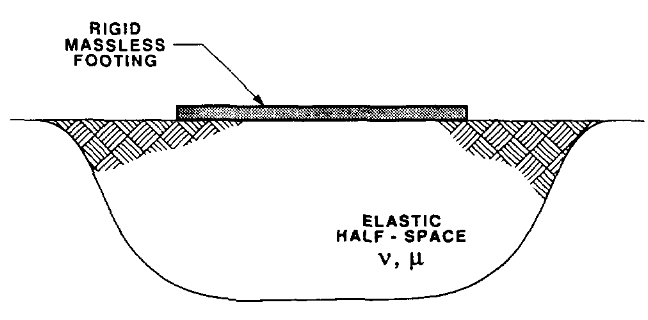

Figure 6. Plane footing on a half-space

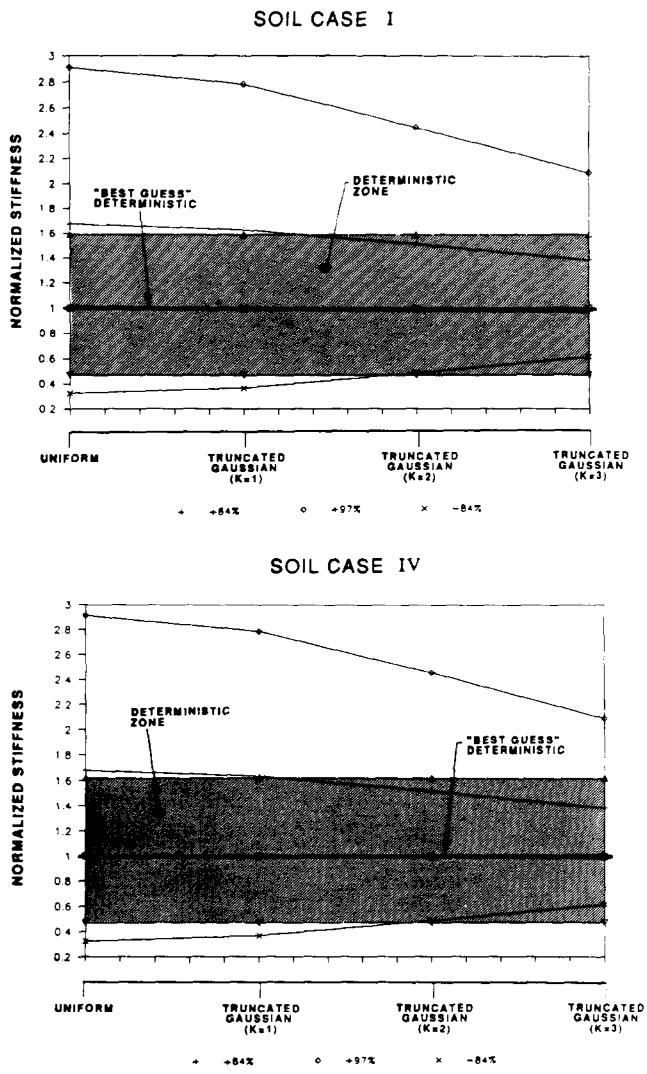

Figure 7. Rocking stiffness on a half-space

of the assumed probability density functions of the material properties is the same as before. The uniform density produces the largest differences from the deterministic zone, while the truncated Gaussian with $k = 3$ produces the closest to the deterministic "best guess".

The behavior of the rocking stiffness with respect to Poisson's ratio is of particular interest. It is clear that for smaller Poisson's ratio, the borders of the deterministic zone are around a non-exceedance probability level of 84%. A non-exceedance level of 97% is always outside the zone. However, for larger Poisson's ratios, the non-exceedance level of 84% shifts upward; thus, the deterministic zone represents an even smaller non-exceedance probability level.

These results mean that if a probability of non-exceedance of 84% is required for the problem at hand (as in many earthquake engineering problems), the deterministic zone approach will be acceptable for smaller Poisson's ratios. For larger Poisson's ratios, the deterministic zone method will result in more flexible springs than required, hence unconservative solutions.

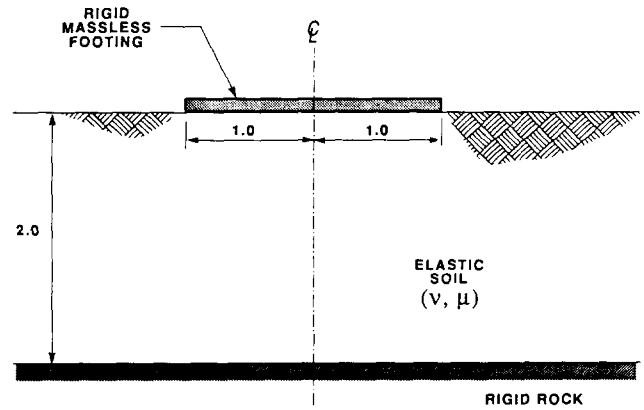

Figure 8. Plane footing on a soil layer (height/width = 1.0)

If a probability of non-exceedance of higher than 84% is required, the deterministic zone approach is always unconservative.

Soil layer with finite depth

Figure 8 shows the plane rigid massless footing resting on a soil layer which is supported by a rigid rock. This system is studied in a fashion similar to the half-space. *Figure 9* shows the rocking stiffness behavior of the footing, while *Figure 10* shows the swaying stiffness of the system, both for cases I and IV.

The rocking stiffness behavior for this case is the same as that of half-space, i.e., the deterministic zone approach is satisfactory for smaller Poisson's ratios and non-exceedance probability levels of 84%. For larger Poisson's ratios or larger non-exceedance probability levels, the deterministic zone approach is not satisfactory, and a probabilistic approach, such as this one, is needed.

The swaying stiffness, *Figure 10,* shows an opposite behavior type. The deterministic zone is much larger. It completely bounds the 84% non-exceedance probability required stiffness for smaller Poisson's ratio. For larger Poisson's ratio, the zone becomes so large that it also envelopes the 97% non-exceedance level. This indicates that the deterministic zone approach requirements may be unnecessarily too demanding, especially for larger Poisson's ratios.

Summary of present work

A probabilistic formulation of plane elastic problems using the boundary element method is introduced. The formulation assumes that both Poisson's ratio and the shear modulus are random variables. Using a Taylor series technique, a perturbation approach is used to solve the problem. The statistical properties of the response measures, such as the mean and mean square are then evaluated. These statistical measures are used to evaluate the response measures at different confidence levels, as required by practicing engineers.

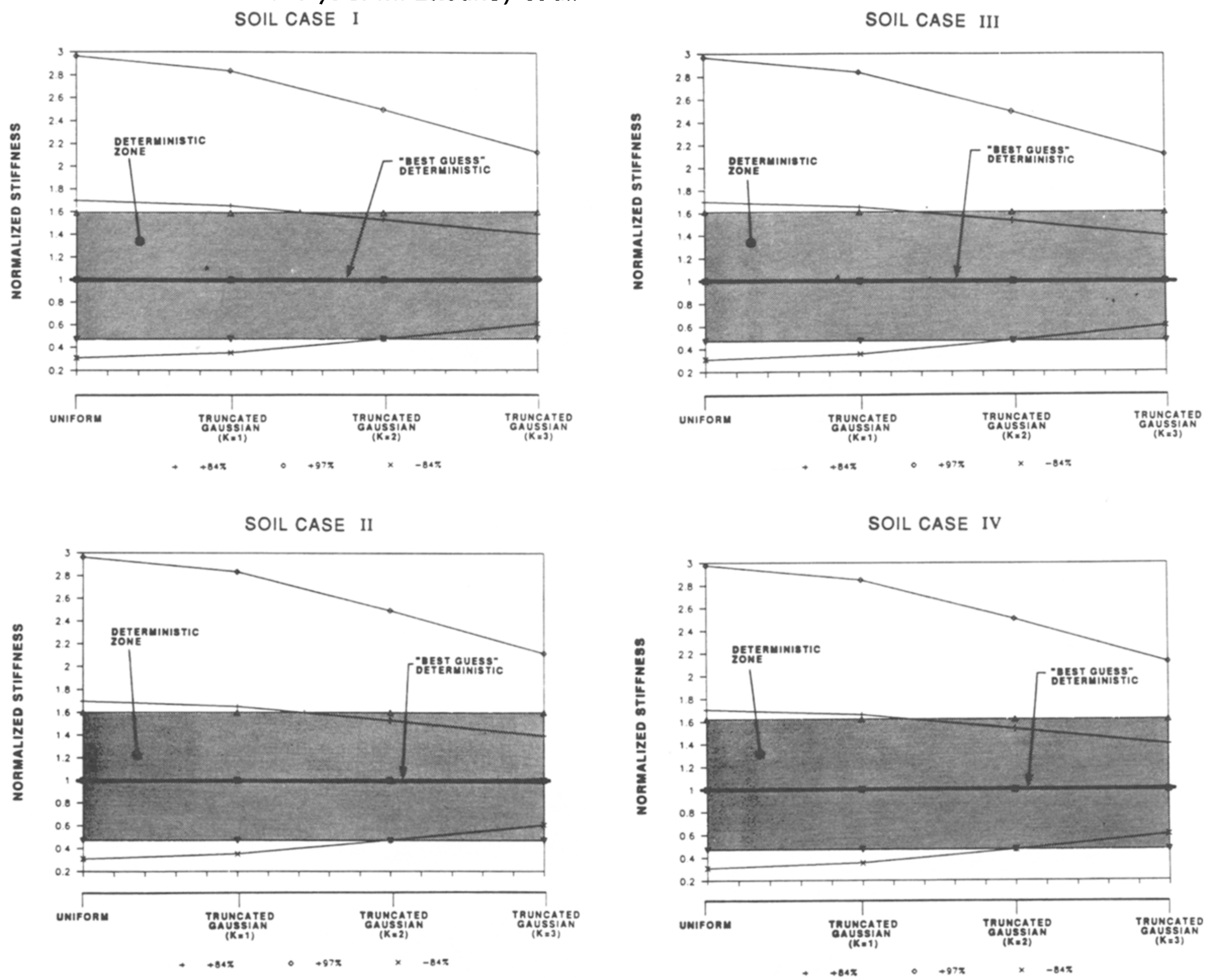

Figure 9. Rocking stiffness (H/B = 2)

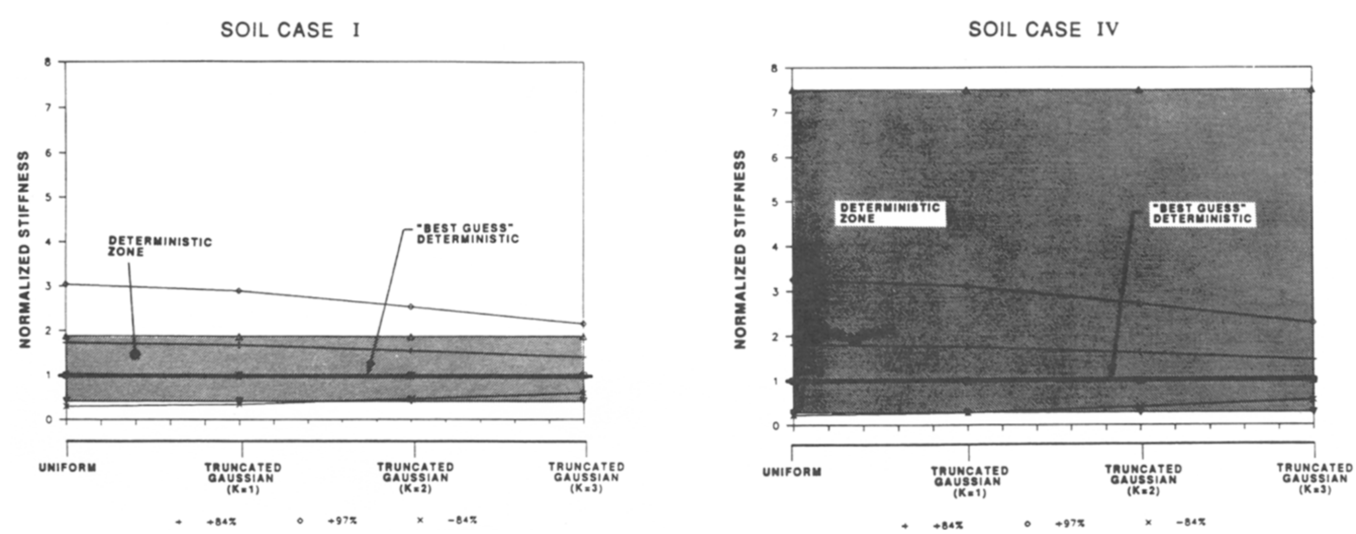

Figure 10. Swaying stiffness (H/B = 2)

Typical and popular structural and soils applications are studied in order to demonstrate the use and value of the proposed method. The shear stiffness of a square beam with random elastic properties, as well as the swaying and rocking stiffnesses of a rigid footing resting on soil, also with random elastic properties, are studied. In both cases, it is established that the probabilistic approach is needed in order to ascertain the conservative solution.

The method introduced is efficient, requiring less than an hour of 286PC–CPU time, thus being a tool that is accessible to the practicing engineer.

We have demonstrated the feasibility and practicality of this approach. It has been presented as an analysis and design tool, since we believe that an analytical method without close coupling to simple and clear design usage is incomplete. We have tried to demonstrate such usage with our applications.

PBEM are still in their infancy. Much remains to be done in order that the tremendous potential of PBEM as a powerful analysis and design tool is fulfilled. Ours is only a first step which will lead to the consideration of three-dimensional static problems, and steady-state dynamic problems.

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